Gaussian Mixture Implementation of the Cardinalized Probability Hypothesis Density Filter for Superpositional Sensors

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***Abstract*—In the past few years, probability hypothesis density (PHD) and cardinalized probability hypothesis density (CPHD) filters have been successfully applied to multi object localization. However, one mayor limitation of the filters originates from the fact that a measurement vector $\mathbf{z}$ may only contain information about a single object. Thus, when localizing multiple objects, a finite set of measurement vectors $\mathcal{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ must be provided, which in the best case contains one measurement vector for each object in the surveillance area. Unfortunately, most sensors do not provide multiple measurements. Instead, preprocessing techniques are often used to subdivide the raw measurement signal into multiple signals. Unfortunately, this is often not easy and results in an information loss.**

Realizing, that there is a need for a CPHD filter variant that does not possess the aforementioned limitation, Mahler provided the theoretical foundation for a CPHD filter variant for superpositional sensors where $\mathbf{z} = \eta(\mathbf{x}_1) + \ldots + \eta(\mathbf{x}_n)$. However, in its current form the CPHD filter for superpositional sensors (SPS-CPHD) is computationally intractable. In this paper, a closed form solution of the SPS-CPHD filter equations using Gaussian mixture models is presented.

***Index Terms*—Multi object localization and tracking, Random finite sets, Superpositional sensors, Probability hypothesis density filter, Gaussian mixture, Closed form CPHD filter, Analytic implementation CPHD filter**

I. Introduction

Ever since the probability hypotheses density (PHD) and the cardinalized probability hypotheses density (CPHD) filters were introduced by Mahler in [1], [2], they have gained popularity. Since then, the CPHD filter was applied to many multi-object detection and tracking problems with promising results. For instance, a sequential Monte Carlo implementation of the PHD filter [3] was used in thermal infrared localization (ThILo) to localize an unknown and time-varying number of persons in home environments [4]. Additionally, a Gaussian mixture (GM) CPHD filter was used in combination with auditory processing to improve multi object speech based localization [5].

Despite the good results that were achieved, the CPHD filter has still a major drawback which originates from one of the assumptions made in its derivation. It is assumed that each measurement vector $\mathbf{z}$ is generated by only one object $\mathbf{x}$ [2]. There are sensors where this assumption is true but in general it is not. In most cases, sensors generate only one measurement which originates from multiple objects [6, p. 5].

Although in some cases special algorithms can be used to extract a finite set of measurements $\mathcal{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$, but this is error prone and not always applicable.

To circumvent this particular limitation, Mahler published the theoretical foundation for a CPHD filter designed for arbitrary superpositional sensors (SPS) [6]. Though the principal equations for superpositional sensors are provided, they are computationally intractable in their current form. The purpose of this paper is to provide a computationally tractable implementation of the SPS-CPHD filter by using GM models.

The organization of this paper is as follows: At first, a basic definition of superpositional sensors is given in section II. Subsequently, the theoretical results of the CPHD filter for superpositional sensors derived in [6] are summarized in section III. Finally, the new SPS-GM-CPHD filter equations for superpositional sensors are presented in section IV. Sections V and VI conclude the publication with some remarks about the computational efficiency of the filter and its applicability to real world problems.

II. Background

In this section, a short explanation of the proposed multi object single measurement model is given and the differences to the standard multi object multi measurement model are highlighted.

A. Multi object multi measurement model

The single object single measurement model has the general form

$$\mathbf{z} = \eta(\mathbf{x}) + \mathbf{w} \tag{1}$$

where $\mathbf{x}$ is the state of a single object, $\mathbf{w}$ is a zero mean random noise vector and the function $\eta(\mathbf{x})$ transforms the state vector $\mathbf{x}$ into the measurement vector $\mathbf{z}$. Assuming, there are no missed detections or false measurements and given the finite set $\mathcal{X}$ of n objects

$$\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}, \tag{2}$$

the finite set $\mathcal{Z}$ of all measurements $\mathbf{z}_i$ is

$$\begin{aligned}\mathcal{Z} &= \{\mathbf{z}_1, \ldots, \mathbf{z}_n\} \\ &= \{\eta(\mathbf{x}_1) + \mathbf{w}_1, \ldots, \eta(\mathbf{x}_n) + \mathbf{w}_n\}.\end{aligned} \tag{3}$$

B. Multi object single measurement model for superpositional sensors

In contrast to the measurement model presented in section II-A which is used in the standard CPHD filter, the multi object single measurement model for superpositional sensors has the form

$$\mathbf{z} = \sum_{\mathbf{x}\in\mathcal{X}} \eta(\mathbf{x}) + \mathbf{w}. \tag{4}$$

Based on measurement equation (4), it can be shown that the multi object likelihood is [6, eqn. 11]

$$f(\mathbf{z}|\mathcal{X}) = f(\mathbf{z} - \sum_{\mathbf{x}\in\mathcal{X}} \eta(\mathbf{x})). \tag{5}$$

III. CPHD FILTER FOR SUPERPOSITIONAL SENSORS

The full derivation of the CPHD filter for superpositional sensors does not fall within the scope of this paper but can be found in [6]. In this section, only the main results are summarized. Since the predictor equations of the CPHD and SPS-CPHD filter do not differ, the reader is referred to [2] for a formal definition of the CPHD predictor equation.

A. Assumptions

For the derivation of the SPS-CPHD corrector equation a multi object likelihood according to equation (5) is assumed. In addition, the predicted or *a priori* multi object distribution $f^-(\mathcal{X}|\mathcal{Z})$ is supposed to be an independent and identically distributed (i.i.d.) cluster process [6, eqn. 25]

$$f^-(\mathcal{X}|\mathcal{Z}) \approx n!\ \ p^-(n) \prod_{\mathbf{x}\in\mathcal{X}} s^-(\mathbf{x}) \tag{6}$$

where $n = |\mathcal{X}|$ is the cardinality of the finite set $\mathcal{X}$, $p^-(n)$ is the *a priori* cardinality distribution and $s^-(x)$ is the *a priori* intensity distribution. In this context, the intensity distribution $s^-(x)$ is equal to

$$s^-(\mathbf{x}) = \frac{D^-(\mathbf{x}|\mathcal{Z}^k)}{\int D^-(\mathbf{x}|\mathcal{Z}^k)\ d\mathbf{x}} \tag{7}$$

where $D^-(\mathbf{x}|\mathcal{Z})$ is the *a priori* probability hypotheses density (PHD).

B. Basic definitions

In contrast to the PHD filter, which only propagates the PHD $D(\mathbf{x}|\mathcal{Z})$, the CPHD filter also propagates the full cardinality distribution $p(n)$. Beginning with the corrector equation for the probability generating functional (p.g.fl.)

$$G^+[h] = \frac{\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(h(\mathbf{x}))]^{\star n})(\mathbf{z})}{\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}\ s^-(\mathbf{x}) \tag{8}$$

as well as for the probability generating function (p.g.f)

$$\begin{aligned} G^+(\mathbf{x}|\mathcal{Z}) &= [G^+[h]]_{h=\mathbf{x}} \\ &= \frac{\sum_{n=0}^{\infty} \mathbf{x}^n\ p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}{\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})} \end{aligned} \tag{9}$$

presented in [6, eqn. 55 and 63] the PHD and the cardinality distribution can be derived. Thereby, the terms

$$(f_a \star f_b)(\mathbf{x}) = \int f_a(\mathbf{y}) f_b(\mathbf{x}-\mathbf{y}) d\mathbf{x} \tag{10}$$

$$f_a^{\star n}(\mathbf{x}) = f_a(\mathbf{x}) \star f_a^{\star(n-1)}(\mathbf{x}) \tag{11}$$

$$f_a^{\star 0}(\mathbf{x}) = \delta_0(\mathbf{x}) \tag{12}$$

define the convolution and n-fold recursive convolution of $f_a(\mathbf{x})$ and $f_b(\mathbf{x})$. Aside from that, $\eta[s^-(\mathbf{x})]$ is defined as

$$\eta[s^-(\mathbf{x})] = \int s^-(\mathbf{x})\ \delta_{\eta(\mathbf{x})}(\mathbf{z})\ d\mathbf{x} \tag{13}$$

where $\delta_{\eta(\mathbf{x})}(\mathbf{z})$ is the Dirac delta function concentrated at $\eta(\mathbf{x})$.

The relationship between the PHD $D^+(\mathbf{x}|\mathcal{Z})$, the cardinality distribution $p^+(n|\mathcal{Z})$ and the expected number of objects $N^+(\mathcal{Z})$ is

$$D^+(\mathbf{x}|\mathcal{Z}) = \frac{\delta G^+}{\delta \mathbf{x}}[1] \tag{14}$$

$$p^+(n|\mathcal{Z}) = \frac{1}{n!} G^{+(n)}(0) \tag{15}$$

$$N^+(\mathcal{Z}) = G^{+(1)}(1) \tag{16}$$

where $G^{+(n)}(x)$ is the n-th derivate of the p.g.f. $G^+(x)$.

C. CPHD corrector equations for superpositional sensors

Under the assumption that the measurement set $\mathcal{Z}$ only consists of a single measurement $\mathcal{Z} = \{\mathbf{z}\}$, the SPS-CPHD corrector equation for the PHD $D^+(\mathbf{x}|\{\mathbf{z}\})$ results in

$$\begin{aligned} &D^+(\mathbf{x}|\{\mathbf{z}\}) = \dots \\ &\frac{\sum_{n=1}^{\infty} n\ p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n-1})(\mathbf{z}-\eta(\mathbf{x}))}{\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}\ s^-(\mathbf{x}). \end{aligned} \tag{17}$$

Furthermore, the cardinality distribution $p^+(n|\{z\})$ corrector equation becomes

$$p^+(n|\{\mathbf{z}\}) = \frac{p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}{\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}. \tag{18}$$

Finally, the estimated number of objects $N^+(\{\mathbf{z}\})$ can be calculated from

$$N^+(\{z\}) = \frac{\displaystyle\sum_{n=1}^{\infty} n\ p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}{\displaystyle\sum_{n=0}^{\infty} p^-(n)\ (f \star \eta[s^-(\mathbf{x})]^{\star n})(\mathbf{z})}. \tag{19}$$

After presenting the most interesting results from [6] in this section, the next section will focus on an analytic implementation of the SPS-CPHD corrector equation with Gaussian mixture models.

IV. GM CPHD FILTER FOR SUPERPOSITIONAL SENSORS

Based on the form of the SPS-CPHD corrector equation presented in section III, a closed form solution of the equation is derived for the special case of linear Gaussian models.

A. Assumptions

Assuming, that the multi object single measurement model is linear

$$\mathbf{z} = \sum_{\mathbf{x}\in\mathcal{X}} \mathbf{H}\,\mathbf{x} + \mathbf{w} \tag{20}$$

with an additive zero mean Gaussian noise vector $\mathbf{w}$ and $\eta(\mathbf{x}) = \mathbf{H}\,\mathbf{x}$ is a linear transformation. Then the multi object likelihood from equation (5) becomes

$$\begin{aligned} f(\mathbf{z}|\mathcal{X}) &= \mathcal{N}\left(\mathbf{z} - \sum_{\mathbf{x}\in\mathcal{X}} \mathbf{H}\,\mathbf{x}; 0, \mathbf{R}\right) \\ &= \mathcal{N}\left(\mathbf{z}; \sum_{\mathbf{x}\in\mathcal{X}} \mathbf{H}\,\mathbf{x}, \mathbf{R}\right) \end{aligned} \tag{21}$$

where $\mathbf{R}$ is the variance of the noise vector $\mathbf{w}$.

Furthermore, it is assumed that the predicted PHD $D^-(\mathbf{x})$ is a Gaussian mixture of the form

$$D^-(\mathbf{x}) = \sum_{j=1}^{J^-} w_j^-\ \mathcal{N}\left(\mathbf{x}; \mathbf{m}_j^-, \mathbf{P}_j^-\right) \tag{22}$$

where $\mathbf{J}$ denotes the number of mixture components and w, $\mathbf{m}$ and $\mathbf{P}$ are their corresponding weights, means and covariances. According to equation (7), this results in the intensity distribution

$$s^-(\mathbf{x}) = \frac{\displaystyle\sum_{j=1}^{J^-} w_j^-\ \mathcal{N}\left(\mathbf{x}; \mathbf{m}_j^-, \mathbf{P}_j^-\right)}{\displaystyle\sum_{j=1}^{J^-} w_j^-}. \tag{23}$$

Finally, the cardinality of the measurement set $\mathcal{Z}$ is assumed to be always $|\mathcal{Z}| = 1$, since the sensor is expected to generate one single measurement vector $\mathbf{z}$. Therefore, the measurement set $\mathcal{Z}$ becomes

$$\mathcal{Z} = \{\mathbf{z}\} \tag{24}$$

whatever of the number n of objects in the surveillance area are.

B. Closed form solution of the CPHD filter for superpositional sensors

Before going into more detail, it is noted that an analytic implementation of the GM-CPHD predictor and corrector equations for the single object single measurement model described in section II-A is available in [7]. Since the measurement model does not change the GM-CPHD predictor equations, those will not be presented in this paper.

Now, suppose that all aforementioned assumptions are satisfied and that the *a priori* intensity function $s^-(\mathbf{x})$ is a Gaussian mixture. Then it can be shown that the *a posteriori* PHD $D^+(\mathbf{x}|\,\{\mathbf{z}\})$ is also a Gaussian mixture of the form

$$D^+(\mathbf{x}|\,\{\mathbf{z}\}) = \dots$$

$$\frac{\displaystyle\sum_{n\geq 1}\sum_{j\in\mathcal{S}}\sum_{\mathrm{I}\in\mathcal{S}^{n-1}} n\ \Psi_{(I\times\{j\})}(n,\mathbf{z})\ \mathcal{N}\left(\mathbf{x}; m_{\mathrm{I}}^{(j)}, P_{\mathrm{I}}^{(j)}\right)}{\displaystyle\sum_{n\geq 0}\ \sum_{\mathrm{I}\in\mathcal{S}^n} \Psi_{\mathrm{I}}(n,\mathbf{z})} \tag{25}$$

where $\mathcal{S}$ is a subset of the natural numbers $\mathbb{N}$, in particular

$$\mathcal{S} = \{j \mid 1 \leq j \leq J^-\} \tag{26}$$

and $\mathcal{S}^n$ is the n-fold set product defined as

$$\mathcal{S}^n = \mathcal{S} \times \ldots \times \mathcal{S} \tag{27}$$

$$= \{(s_1, \ldots, s_n) \mid s_1, \ldots, s_n \in \mathcal{S}\} \tag{28}$$

$$\mathcal{S}^0 = \emptyset. \tag{29}$$

As a result, every element $\mathrm{I} \in \mathcal{S}^n$ is a tuple of indices.

Furthermore, the individual terms are

$$\Psi_{\mathrm{I}}(n,\mathbf{z}) = p^-(n)\ w_{\mathrm{I}}\ q_{\mathrm{I}}(\mathbf{z}) \tag{30}$$

$$q_{\mathrm{I}}(z) = \mathcal{N}\left(\mathbf{z}; \mu_{\mathrm{I}}, \mathbf{\Sigma}_{\mathrm{I}}\right) \tag{31}$$

$$\mu_{\mathrm{I}} = \sum_{i\in\mathrm{I}} \mathbf{H}\,\mathbf{m}_i^- \tag{32}$$

$$\mathbf{\Sigma}_{\mathrm{I}} = \sum_{i\in\mathrm{I}} \mathbf{H}\,\mathbf{P}_i^-\ \mathbf{H}^{\mathrm{T}} + \mathbf{R} \tag{33}$$

$$w_{\mathrm{I}} = \frac{\prod_{i\in\mathrm{I}} w_i^-}{\sum_{j=1}^{J^-} w_j^-} \tag{34}$$

and the new means and covariances are

$$\mathbf{m}_{\mathrm{I}}^{(j)} = \mathbf{m}_j^- + \mathbf{K}_{\mathrm{I}}^{j}\left(\mathbf{z} - \mathbf{H}\,\mathbf{m}_j^- - \mu_{\mathrm{I}}\right) \tag{35}$$

$$\mathbf{P}_{\mathrm{I}}^{(j)} = \left[\mathbf{E} - \mathbf{K}_{\mathrm{I}}^{(j)}\,\mathbf{H}\right]\ \mathbf{P}_j^- \tag{36}$$

$$\mathbf{K}_{\mathrm{I}}^{(j)} = \mathbf{P}_j^-\ \mathbf{H}^T\ \left[\mathbf{H}\,\mathbf{P}_j^-\ \mathbf{H}^{\mathrm{T}} + \mathbf{\Sigma}_{\mathrm{I}}\right]^{-1} \tag{37}$$

where $\mathbf{E}$ is the identity matrix.

Finally, the corrector equation for the *a posteriori* cardinality distribution $p^+(n|\{\mathbf{z}\})$ is then

$$p^+(n|\{\mathbf{z}\}) = \frac{\sum_{\mathrm{I}\in\mathcal{S}^n} \Psi_{\mathrm{I}}(n,\mathbf{z})}{\sum_{n\geq 0}\sum_{\mathrm{I}\in\mathcal{S}^n} \Psi_{\mathrm{I}}(n,\mathbf{z})} \tag{38}$$

and the estimated number of objects $N^+(\{\mathbf{z}\})$ are

$$N^+(\{\mathbf{z}\}) = \frac{\sum_{n\geq 1}\sum_{\mathrm{I}\in\mathcal{S}^n} n\ \Psi_{\mathrm{I}}(n,\mathbf{z})}{\sum_{n\geq 0}\sum_{\mathrm{I}\in\mathcal{S}^n} \Psi_{\mathrm{I}}(n,\mathbf{z})}. \tag{39}$$

It is noted, that a complete derivation of the filter equations is not within the scope of this shortpaper due to space limitation. However, it will be given later in the full paper in conjuction with numerical examples.

V. Implementation issues

Despite the fact, that the proposed GM approximation results a closed form solution, the computational complexity is still very high. In theory, the cardinality distribution may be infinitely tailed, resulting in an infinite sum over all combinations. However, in practice a minimum and maximum number of possible objects N_{min} and N_{max} may be defined to reduce the computational complexity. Nevertheless, one must consider that all possible combinations including duplicates of the GM components of length n must be calculated, thus resulting in

$$J^+ = \sum_{N_{min}\leq n\leq N_{max}} \binom{n+J^- - 1}{J^-} \tag{40}$$

new GM components. Therefore, it is crucial to reduce the number of mixture components to a minimum after each correction step. However, several approaches to reduce GM components exist. For instance, a simple pruning and merging mechanism [8] could be used or even more advanced methods [9] may be applied.

VI. Remarks

In this paper, only the resulting SPS-GM-CPHD filter equations for superpositional sensors were presented. Nevertheless, a formal prove exists but is not part of the paper due to space limitation. The same is true for the numerical studies, but both will be delivered in a future publication. Although, these filter equations are only directly applicable for linear Gaussian sensors models, nonlinear Gaussian approximations [7] can be used for mildly nonlinear prediction and sensor models. Additionally, the computational complexity is still too high to be usable in scenarios with more than a few objects. But still, when the number of objects is small and the number of GM components is kept small, the filter may be applicable to real world problems. However, it is important to further reduce the computational complexity.

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